Extension bundles and the standard model

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ABSTRACT: We construct a new class of stable vector bundles suitable for heterotic string compactifications. Using these we describe a novel way to derive the fermionic matter content of the Standard Model from the heterotic string. More precisely, we can get either the Standard Model gauge group G_{SM} times an additional $\mathrm{U}(1)$, or just G_{SM} but with additional exotic matter. For this we compactify on an elliptically fibered Calabi-Yau threefold X with two sections, the B-fibration, a variant of the ordinary Weierstrass fibration, which allows X to carry a free involution. We construct rank five vector bundles, invariant under this involution, such that turning on a Wilson line we obtain the Standard Model gauge group and find various three generation models. This rank five bundle is derived from a stable rank four bundle that arises as an extension of bundles pulled-back from the base and twisted by suitable line bundles. We also give an account of various previous results and put the present construction into perspective.

KEYWORDS: Differential and Algebraic Geometry, Superstrings and Heterotic Strings.

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Contents

1. Introduction

The goal of the present paper is to get a (supersymmetric) phenomenological spectrum from the $E_8 \times E_8$ heterotic string on a Calabi-Yau space X. More precisely, one wants to construct a model leading in four dimensions to the gauge group and net chiral matter content of the Standard Model. The individual number of generations and anti-generations and the number of Higgs multiplets will be investigated elsewhere.

A common method to get the Standard Model gauge group G_{SM} is to have first a GUT gauge group $H = SU(5)$ from embedding a vector bundle of structure group $G = SU(5)$ into the first E_8. Then, if $\pi_1(X) = \mathbf{Z_2}$, one uses a Wilson line to break H to G_{SM}.

We will choose X to be elliptically fibered over $B = \mathbf{F_k}$; X is smooth for $k = 0, 1, 2$. We choose a specific fibration type (the B-fibration) which has two sections σ_1, σ_2. On this X we find a free involution τ_X (as needed because $\pi_1(X) = 0$) for k even. We find invariant bundles of ± 6 generations to get the Standard Model on the quotient $X' = X/\mathbf{Z_2}$.

The vector bundle on X will be constructed using the method of bundle extensions investigated in [1], cf. also [2]. More precisely, we consider extension bundles of rank $n+m$ defined by

$$0 \to \pi^* E_n \otimes \mathcal{O}_X(-mD) \to V_{n+m} \to \pi^* E_m \otimes \mathcal{O}_X(nD) \to 0 \qquad (1.1)$$

with $D = x\Sigma + \pi^*\alpha$ (where $\Sigma := \sigma_1 + \sigma_2$) and E_i vector bundle on the base of vanishing first Chern class which are stable with respect to a Kähler class H on B. We will show that V_{n+m} is stable with respect to the Kähler class $J = z\Sigma + \pi^*H$ for the real number $z > 0$ in a suitable range. One finds that the generation number $N_{gen} = c_3(V_{n+m})/2$ is proportional to x which one therefore has to choose to be non-zero (the actual physical net number of Standard Model generations is computed downstairs on X' and is $N_{gen}/2$).

In order that V_{n+m} qualifies as a physical gauge bundle it has to satisfy the anomaly constraint that requires (cf. below) $W = w_B\Sigma + a_f F = c_2(X) - c_2(V_{n+m})$ to be an effective class (we put here the trivial bundle in the hidden sector). So in particular

$$w_B = \left(6 - \frac{1}{2}nm(n+m)x^2\right)c_1 + nm(n+m)x\alpha \qquad (1.2)$$

has to be an effective curve class in B. As we are interested in rank five vector bundles we find for (n, m) with $(3, 2)$ or $(4, 1)$ the first term in (1.2) to be negative. Moreover, stability of V_{n+m} requires for $x \neq 0$ that $x\alpha H \leq 0$. This implies that the last term in (1.2) also contributes negatively (meaning here: it can not be a non-zero effective class). So the construction (1.1) favors physical gauge bundles of rank four (or less) of type $(n, m) = (2, 2)$ (cf. below; the case $(3, 1)$ is also ruled out by the same w_B not effective argument).

An invariant model on the cover space X with GUT gauge group $SU(5)$ arises actually not from an $SU(5)$ bundle but from an $SU(4) \times U(1)$ bundle (of ± 6 net generations)

$$V_5 = V_4 \otimes \mathcal{O}_X(-\pi^*\beta) \oplus \mathcal{O}_X(4\pi^*\beta) \qquad (1.3)$$

The $U(1)$ of the commutator subgroup $SU(5) \times U(1)$ respresenting the unbroken gauge group in the first E_8 is for $\alpha\beta \neq 0$ anomalous, becomes massive and decouples from the low-energy spectrum. V_5 is poly-stable and invariant; actually $\beta = \pm(1, -1)$.

The invariant rank 4 bundle V_4 of $c_1(V_4) = 0$ is defined by a non-split extension

$$0 \to \pi^* E_1 \otimes \mathcal{O}_X(-D) \to V_4 \to \pi^* E_2 \otimes \mathcal{O}_X(D) \to 0 \qquad (1.4)$$

Here and from now on E_1 and E_2 refer to bundles of rank two on the base.

We will show that the extension can be chosen non-split and V_4 to be invariant. A crucial part of the argument will be to show the stability of V_4 with respect to a Kähler form $J = z\Sigma + \pi^* H$. For a technical reason the Kähler class H on the base has to be chosen to be proportional to $c_1 := c_1(B)$; so we will work finally over $\mathbf{F_0} = \mathbf{P^1} \times \mathbf{P^1}$.

Assuming that the bundles E_i on the base are chosen to be invariant under the base part τ_B of the involution τ_X the pull-back bundles turn out to be invariant as well. The line bundle twist $\mathcal{O}_X(D)$ can be chosen invariant as can the extension bundle V_5.

So the original problem to construct a suitable bundle on X is reduced to construct invariant bundles E_i on the base of suitable instanton numbers $k_i = c_2(E_i)$ (to get $N_{\text{gen}} = \pm 6$). Examples of appropriate base bundles are given in the appendix. Thus we have achieved our goal to construct a heterotic Standard Model compactification. For other heterotic derivations of the Standard Model via an intermediate SU(5) GUT group cf. [3, 4]. The physical generation number $N_{\text{gen}}^{\text{phys}} = N_{\text{gen}}/2$ (downstairs on $X' = X/\mathbf{Z_2}$) is given by

$$N_{\text{gen}}^{\text{phys}} = k_1 - k_2 \qquad (1.5)$$

The E_i are two stable, invariant rank two vector bundles on B of $c_1(E_i) = 0$ and $c_2(E_i) = k_i$. We find that D is given by the invariant divisor $D = \Sigma + \pi^* \alpha$ where α is $(-2, -2)$ or $(-1, -1)$. The list of applicable instanton numbers (k_1, k_2) for the various choices of α is given in table 1 (for $\beta = (1, -1)$; for the negative of that one just has to interchange again the k_i). These data-sets fulfill all necessary conditions for the existence of a non-split extension, stability, DUY-equation, fivebrane effectivity and generation number.

There is a fundamental alternative in this construction: one can either, as described above, cancel unwanted exotic matter (apart from a wellcomed right-handed neutrino ν_R) produced by the decomposition of $\mathbf{248}$ under $SU(4) \times SU(5) \times U(1)_X$; the condition is $\alpha\beta = 0$, thus one keeps the U(1) which is now not anomalous. Or one keeps that matter but avoids the U(1); then the solutions are given in appendix D. (or their reflections (under $(p, q) \to (q, p)$), when at the same time the numbers k_1 and k_2 are interchanged).

Relation to previous bundle constructions. To put the construction presented in this paper into perspective, let us indicate how it has arisen as culmination of previous investigations [5–7, 1]–[8] along similar lines which incorporated different subsets of the whole procedure.

Attempts to get a (supersymmetric) phenomenological spectrum with gauge group G_{SM} and chiral matter content of the Standard model from the $E_8 \times E_8$ heterotic string on a Calabi-Yau space X started with embedding the spin connection in the gauge connection giving an unbroken E_6 (times a hidden E_8 coupling only gravitationally). More generally [9], one can instead of the tangent bundle embed an $G = SU(n)$ bundle for $n = 4$ or 5, leading to unbroken $H = SO(10)$ or $SU(5)$ of even greater phenomenological interest. A concrete description of vector bundles on a general Calabi-Yau space X (not given

via projective embedding) was made in [10] for the case that X has an elliptic fibration $\pi : X {\rightarrow} B$. The net generation number for these bundles was computed in [11, 12]. It was soon realized that the only elliptically fibered X that has non-trivial fundamental group has the Enriques surface as base and leads to generation number zero.

Therefore the following indirect strategy had to be employed. If there is an freely acting group $\mathcal{G}$ on the usually simply-connected X, one can work on $X' = X/\mathcal{G}$ with $\pi_1(X') = \mathcal{G}$ allowing a further breaking of H by turning on Wilson lines. This was achieved in [5] in a general way by using a non-standard elliptic fiber (the B-fibration) which leads to an elliptic fibration of X having two sections σ_1 and σ_2. This led to a free involution τ_X with the required properties.

The necessary invariance of the bundle (so that it descends to the quotient) was checked first [5] only on the level of cohomological invariants (cf. [13] for a similar procedure). Then the related action on the spectral parameters defining the bundle and the corresponding invariance was investigated [6]. Due to an ensuing integrality problem by a factor $1/2$ (essentially the question whether $\Sigma = \sigma_1 + \sigma_2$ can be assumed to restrict to an even class on the spectral surface) the spectral bundle construction itself was adapted more properly to the case of the B-fibration [7].

Although the invariance problem was solved, as a side effect the necessity of a structure group of even rank emerged and also an even number of Standard Model net generations. The first issue is overcome by using an SU(4) bundle V_4 and embedding the U(4) bundle (emerging after a line bundle twist) [14] in E_8 via $V_4 \otimes \mathcal{L} \oplus \mathcal{L}^{-4}$. The rank 5 bundle arising then has to be poly-stable (which in case that $c_1(\mathcal{L}) = y\Sigma + \pi^*\beta$ has non-zero y leads to the necessity of including one-loop effects [7]). Because of the second problem the class of bundles was enlarged [1, 8] by considering also non-split extensions of line bundles by SU(4) bundles, the latter chosen to be pulled back from the base rather than being spectral. There the case of the A-fibration was considered, leading to GUT models, and the Enriques base giving the Standard Model gauge group; the latter case leads to problems when including the condition of effectivity of the fivebrane class from anomaly cancellation.

Using extensions in the case of the B-fibration for spectral or pull-back bundles is possible, but leads again to problems with the effectivity of the fivebrane class. Only when $y = 0$ did these problems disappear, but this also suppresses the chiral generations. Therefore this generation number has to emerge from a different parameter which in fact should be present already in V_4. Hence we employ the more general bundle construction of an extension of a bundle of higher rank (no longer a line bundle) by another bundle, already touched upon in [1, 8]. This is done here for the case of two bundles of rank two, which themselves arise as pullbacks twisted by a line bundle (having $x \neq 0$) as described above. Then here an extension of a line bundle by such a rank 4 bundle is considered.

In section 2 we set the stage and to establish some notation. We recall some generalities of the bundle construction concerning extensions, stability, the physical constraints of effectivity of the fivebrane class and the phenomenological net number of chiral matter generations. In section 3 we review the Calabi-Yau spaces with B-fibration, which have two sections and admit a free involution τ_X. In section 4 we construct the extension described above, show that it can be chosen non-split and that V_5 is stable. In section 5

we describe the way how to get the rank 5 bundle from the rank 4 bundle and various question connected with this procedure. In section 6 we collect all constraints and present some solutions. In section 7 we collect our conclusions. In the appendix we give some useful cohomological formulae and examples of stable and invariant bundles E_i on the base and specify the numerical constraints on the instanton numbers. Further we describe a general argument how to count the number of invariant bundles.

2. Bundles and physical constraints

We begin with some general remarks on the constraints on the bundles used in a heterotic compactification. These concern first the equations of motion of the underlying string theory. Thanks to the work of Donaldson, Uhlenbeck and Yau this can be translated to the mathematical condition of stability. Then we move on to the special case that the bundle V arises as an extension of other bundles. In this case one gets immediately two necessary conditions from the stability of V, in particular in our case a non-split condition for the extension arises.

Up to this point the physical conditions are the same as a pure mathematical investigation would pose, namely the requirement of stability. After this the physical investigation proceeds to pose further requirements: first, one condition of physical consistency (anomaly cancellation, this comes down to the effectivity of the five-brane class); then a phenomenological requirement on the number of Standard model net generations is posed.

2.1 Stability

The main order in constructing a heterotic compactification is to solve the physical equations of motion. For the underlying space this can be reduced to the topological question of constructing a Calabi-Yau space. For the bundle sector one reduces the Kähler-Yang-Mills equations for a G-valoued connection, via the Donaldson-Uhlenbeck-Yau (DUY) theorem, to the construction of a holomorphic vector bundle which has to be stable. Like the Calabi-Yau condition on the underlying space X, the holomorphicity and stability of the vector bundle V are direct consequences of the required four-dimensional supersymmetry. The demand is that a connection A on V has to satisfy the DUY equation

$$F_A^{2,0} = F_A^{0,2} = 0, \quad F_A^{1,1} \wedge J^2 = 0 \tag{2.1}$$

The first equation implies the holomorphicity of V; the second equation is the Hermitian-Yang-Mills (HYM) equation $F_A^{1,1} \wedge J^{n-1} = c \cdot I_F \cdot J^n$ for $n = 3$ with $c \in \mathbf{C}$ vanishing. The latter has, after taking the trace and integrating, the integrability condition [15, 16]

$$\int_X c_1(V) \wedge J^2 = 0 \tag{2.2}$$

This necessary condition becomes sufficient for the existence of a unique solution if V is stable (or, more generally, polystable, i.e., a sum of stable bundles with the same slope). Stability of V (with respect to J) means $\mu_J(V') < \mu(V)$ for all coherent subsheafes V' of V of $rk\, V' \neq 0, rk\, V$ (it suffices to test the V' with V/V' torsion-free, cf. Ch. 4, Lemma 5 [17]). Here $\mu(V) = \frac{1}{rk\, V} \int c_1(V) J^2$ is the slope of V with respect to J.

2.2 Extension bundles and the non-split condition

For a zero-slope bundle V constructed as an extension (with U and W stable)

$$0 \to U \to V \to W \to 0 \qquad (2.3)$$

one finds two immediate conditions which are necessary for stability

(i) $\mu(U) < 0$

(ii) the W of $\mu(W) > 0$ is not a subbundle of V, i.e., the extension (2.3) is non-split

The first condition comes down in our case

$$DJ^2 = 2x(h-z)^2 c_1^2 + 2z(2h-z)\alpha c_1 \; > \; 0 \qquad (2.4)$$

The non-split condition can be expressed as $Ext^1(W,U) = H^1(X, W^* \otimes U) \neq 0$.

2.3 Anomaly constraint and net generation number

Anomaly cancellation forces the three-form field strength H to satisfy $dH = trR \wedge R - TrF \wedge F$ where R and F are the curvature forms of the spin connection on X and the gauge connection on V. This gives the topological condition $c_2(TX) = c_2(V)$. The inclusion of (magnetic) five-branes changes the topological constraint [10] on the gauge bundle V by contributing a source term to the Bianchi identity for the three-form H $dH = trR \wedge R - TrF \wedge F - n_5 \sum_{\text{five-branes}} \delta_5^{(4)}$. The current $\delta_5^{(4)}$ integrates to one in the direction transverse to a five-brane of class $[W]$. Integration over a four-cycle in X gives $c_2(TX) = c_2(V) + [W]$. Supersymmetry requires that five-branes are wrapped on holomorphic curves and $[W]$ has to be an effective class. So the effectivity of $[W]$ constraints the choice of vector bundles V. In the elliptically fibered case $H^4(X)$ decomposes as

$$H^4(X) = \sigma_1 \, H^2(B) \oplus \sigma_2 \, H^2(B) \oplus \pi^* H^4(B) \qquad (2.5)$$

Actually, $c_2(X)$ (cf. below) and $c_2(V)$ (by the invariance requirement) lie in the symmetric subspace $\Sigma \, H^2(B) \oplus \pi^* H^4(B)$ where $\Sigma = \sigma_1 + \sigma_2$. The effectivity condition for the class

$$W = w_B \Sigma + a_f F \qquad (2.6)$$

becomes (here $w_B \geq 0$ means that the class is effective)

$$w_B \geq 0 \, , \quad a_f \geq 0 \qquad (2.7)$$

In general, the decomposition of the ten-dimensional Dirac operator with values in V shows that massless four-dimensional fermions are in one to one correspondence with zero modes of the Dirac operator D_V on X whose index is $\text{index}(D_V) = \sum_{i=0}^3 (-1)^k \dim H^k(X, V) = \int_X Td(X)ch(V)$. For stable vector bundles one has $H^0(X, V) = H^3(X, V) = 0$ and so $\dim H^2(X, V) - \dim H^1(X, V) = \frac{1}{2} \int_X c_3(V)$. For the net number of chiral matter generations one gets with

$$N_{\text{gen}} = h^1(X, V_5^*) - h^1(X, V_5) = \int_X ch(V_5)Td(X) = \int_X \frac{c_3(V_5)}{2} \qquad (2.8)$$

which we want to equal ± 6 in order to get downstairs on $X' = X/\mathbf{Z_2}$ the ± 3 phenomenological net generations of Standard Model fermions.

3. Review of the elliptic Calabi-Yau space with two sections

We consider a Calabi-Yau threefold X, elliptically fibered over a Hirzebruch surface $\mathbf{F}_m$, whose generic fiber is described by the so-called B-fiber $\mathbf{P}_{1,2,1}(4)$ instead of the usual A-fiber $\mathbf{P}_{2,3,1}(6)$ (the subscripts indicate the weights of x, y, z). X is given by a generalized Weierstrass equation which embeds X in a weighted projective space bundle over $\mathbf{F_m}$

$$y^2 + x^4 + a_2 x^2 z^2 + b_3 x z^3 + c_4 z^4 = 0 \tag{3.1}$$

where x, y, z and a, b, c are sections of K_B^{-i} with $i = 1, 2, 0$ and $i = 2, 3, 4$, respectively.

X admits two cohomologically inequivalent section σ_1, σ_2. For this consider (3.1) at the locus $z = 0$, i.e., $y^2 = x^4$ (after $y \to iy$). One finds 8 solutions which constitute the two equivalence classes $(x, y, z) = (1, \pm 1, 0)$ in $\mathbf{P}_{1,2,1}$. We choose $y = +1$, corresponding to the section σ_1, as zero in the group law, while the other one can be brought, for special points in the moduli space, to a half-division point (in the group law) leading to the shift-involution. Let us keep on record the relation of divisors (with $\sigma_i := \sigma_i(B)$, $i = 1, 2$)

$$(z) = \Sigma := \sigma_1 + \sigma_2, \quad \sigma_1 \cdot \sigma_2 = 0 \tag{3.2}$$

The fibration structures leads in the following way to the cohomological data of Z (unspecified Chern classes like c_1 refer to the base B; further we write $c_i = \pi^* c_i(B)$).

As noted z, x, y can be thought of as homogeneous coordinates on a $\mathbf{P}_{1,2,1}$ bundle W, i.e. as sections of line bundles $\mathcal{O}(1), \mathcal{O}(1) \otimes \mathcal{L}$ and $\mathcal{O}(1)^2 \otimes \mathcal{L}^2$ whose first Chern classes are given by r, $r + c_1$, $2r + 2c_1$ with $c_1(\mathcal{O}(1)) = r$. The cohomology ring of W is generated by r with the relation $r(r + c_1)(2r + 2c_1) = 0$ expressing the fact that z, x, y have no common zeros. As the B-model is defined by the vanishing of a section of $\mathcal{O}(1)^4 \otimes \mathcal{L}^4$, which is a line bundle over W with first Chern class $4(r + c_1)$, the restriction from W to X is effected by multiplying by this Chern class, so that $c(W) = (1 + 4r + 4c_1)c(X)$. One can then simplify $r(r + c_1)(2r + 2c_1) = 0$ to $r(r + c_1) = 0$ in the cohomology ring of X and finds

$$c(X) = c(B) \frac{(1 + r)(1 + r + c_1)(1 + 2r + 2c_1)}{1 + 4r + 4c_1} \tag{3.3}$$

With $r^2 = -rc_1$ and the class $r = \sigma_1 + \sigma_2$ (as $z = 0$ implies $y^2 = x^4$ giving $(x, y) = (i, 1)$ and $(i, -1)$) of the divisor $(z = 0)$ of the section z of the line bundle $\mathcal{O}(1)$ we find

$$c_2(X) = \pi^* c_2 + 6\Sigma \pi^* c_1 + 5\pi^* c_1^2, \quad c_3(X) = -36\pi^* c_1^2 \tag{3.4}$$

From the weights a_2, b_3 and c_4 of the defining equation one gets $5^2 + 7^2 + 9^2 - 3 - 3 - 1 = 148$ complexe structure deformations over $\mathbf{F_0}$. This is consistent with the Euler number and the $h^{1,1}(X) = 4$ Kähler classes

$$h^{1,1}(X) = 4, \quad h^{2,1}(X) = 148, \quad \text{and} \quad e(X) = -288 \tag{3.5}$$

For later use let us also note the adjunction relations

$$\sigma_i^2 = -\pi^* c_1 \sigma_i, \quad \Sigma^2 = -\pi^* c_1 \Sigma \tag{3.6}$$

The Kähler cone. For the base B being given by a Hirzebruch surface $\mathbf{F_m}$ (with $m = 0, 2$) $H^2(B, \mathbf{Z})$ generated by the effective base and fiber classes b and f (with intersection relations $b^2 = -m$, $b \cdot f = 1$ and $f^2 = 0$). Obviously, these two classes represent actual curves. The effective cone (non-negative linear combinations of classes of actual curves) is given by the condition $p \geq 0, q \geq 0$ on $\rho = pb + qf$; this we denote by $\rho \geq 0$. The Kähler cone $\mathcal{C}_B$ of B (where $\rho \in \mathcal{C}_B$ means $\rho\zeta > 0$ for all actual curves of classes ζ or equivalently $\rho b > 0, \rho f > 0$) is given by $\mathcal{C}_B = \{t_1 b^+ + t_2 f | t_i > 0\}$ (with $b^+ = b + mf$). For example on $\mathbf{F_2}$ one has $c_1 \notin \mathcal{C}_B$ as $c_1 b = 0$.

Let $J = x_1\sigma_1 + x_2\sigma_2 + \pi^*H$ be an element in the Kähler cone $\mathcal{C}_X$ ($H \in \mathcal{C}_B$). Demanding that its intersections with the curves F and $\sigma_i\alpha$ are non-negative amounts to $x_1 + x_2 > 0$ and $(H - x_i c_1)\alpha > 0$. Similarly intersecting J^2 with σ_i and α gives the conditions $(H - x_i c_1)^2 > 0$ and $(2\sum x_i\ H - \sum x_i^2\ c_1)\alpha > 0$. Integrating J^3 gives $\sum x_i\ (H - x_i c_1)^2 + (2\sum x_i\ H - \sum x_i^2\ c_1)H > 0$. From this one gets the condition for J to be ample (positive)

$$J = x_1\sigma_1 + x_2\sigma_2 + \pi^*H \in \mathcal{C}_X \quad \Longleftrightarrow \quad x_1 + x_2 > 0 \quad, \quad H - x_i c_1 \in \mathcal{C}_B \qquad (3.7)$$

Concretely we will choose $J = z\Sigma + \pi^* h c_1$ giving the condition $0 < z < h$. Below we will restrict to the case $B = \mathbf{F_0}$ as we will have to use the fact that $c_1 \in \mathcal{C}_B$.

3.1 Existence of a free $\mathbf{Z}_2$ operation

We give a free involution τ_X on X which leaves the holomorphic three-form invariant; then $X' = X/\tau_X$ is a smooth Calabi-Yau. We assume τ_X compatible with the fibration, i.e., we assume the existence of an involution τ_B on the base B with $\tau_B \cdot \pi = \pi \cdot \tau_X$.

We will choose for τ_B the following operation in local (affine) coordinates

$$b = (z_1, z_2) \quad \xrightarrow{\tau_B} \quad -b = \tau_B(b) = (-z_1, -z_2) \qquad (3.8)$$

To define τ_X one combines τ_B with an operation on the fibers (cf. [5, 6]). A free involution on a smooth elliptic curve is given by translation by a half-division point. Such an object has to exist globally; this is the reason we have chosen the B-fibration where X possesses a second section. If we would tune $\sigma_2(b) \in E_b$ to be a half-division point the condition $b_3 = 0$ would ensue and X would become singular. Therefore this idea has to be enhanced. Furthermore, even for a B-fibered X those fibers lying over the discriminant locus in the base will be singular where the freeness of the shift might be lost. As the fixed point locus of τ_B is a finite set of (four) points we can assume that it is disjoint from the discriminant locus (so points in the singular fibers are still not fixed points of τ_X).

One finds [5] as τ_X over $\mathbf{F_m}$ with m even (i.e., m being 0 or 2) the free involution

$$(z_1, z_2\,; x, y, z) \xrightarrow{\tau_X} (-z_1, -z_2\,; -x, -y, z) \qquad (3.9)$$

This exchanges the points $\sigma_1(b) = (b\,; 1, 1, 0)$ and $\sigma_2(-b) = (-b\,; 1, -1, 0)$ between the fibers E_b and $E_{-b} = E_{\tau_B(b)}$; in $\mathbf{P}_{1,2,1}$ the sign in the x-coordinate can be scaled away here in contrast to the sign in the y-coordinate. As indicated above an involution like in (3.9) could not exist on the fiber alone, i.e. as a map $(x, y, z) \longrightarrow (-x, -y, z)$, because this would force one, from (3.1), to the locus $b_3 = 0$ where X becomes singular (so only

then is this defined on the fiber and so, being a free involution, a shift by a half-division point). But it can exist combined with the base involution τ_B on a subspace of the moduli space where the generic member is still smooth: from (3.1) the coefficient functions should transform under τ_X as a_2^+, b_3^-, c_4^+, so over $\mathbf{F_0}$ only monomials $z_1^p z_2^q$ within $b_{6,6}$ with $p+q$ even (in $a_{4,4}$ and $c_{8,8}$ with $p+q$ odd) are forbidden. So the number of deformations drops to $h^{2,1}(X) = (5^2+1)/2 + (7^2-1)/2 + (9^2+1)/2 - 1 - 1 - 1 = 75$. The discriminant remains generic as enough terms in a, b, c survive, so Z is still smooth, cf. [5]. The Hodge numbers $(4, 148)$ and $(3, 75)$ of X and X' show that indeed $e(X') = e(X)/2$ (X' has lost one divisor as the two sections are identified).

4. Stable SU(4) bundles on X

Let E_1 and E_2 be two stable rank two vector bundles on B of $c_1(E_i) = 0$ and $c_2(E_i) = k_i$. We consider the extension defining our rank four bundle (with $D = x\Sigma + \pi^*\alpha$)

$$0 \to \pi^* E_1 \otimes \mathcal{O}_X(-D) \to V_4 \to \pi^* E_2 \otimes \mathcal{O}_X(D) \to 0 \qquad (4.1)$$

4.1 Stability of $\pi^* E_i$

We prove that $\pi^* E$ is stable on X with respect to a $J = z\Sigma + \pi^* H$ in the Kähler cone $\mathcal{C}_X$ (i.e. $H - zc_1 \in \mathcal{C}_B$ [7], so $z < h$) if E of $c_1(E) = 0$ is stable on B with respect to $H = hc_1$. Thus, for $c_1 \in \mathcal{C}_B$, we assume from now on that $B = \mathbf{F_0}$. Following Lemma 5.1 [1] let $\mathcal{F}$ be a subsheaf of $\pi^* E$ where we can assume that $\pi^* E/\mathcal{F}$ is torsion free (cf. Ch. 4, Lemma 5 [17]); we have $0 \to \mathcal{F}|_{\sigma_i} \to E$ and $c_1(\mathcal{F}|_{\sigma_i})H < 0$. Similarly we get $0 \to \mathcal{F}|_F \to \mathcal{O}_F^r$ thus $deg(\mathcal{F}|_F) \leq 0$ as $\mathcal{O}_F^r$ is semistable (where $r := rk(E)$). Then for $H - zc_1 \in \mathcal{C}_B$ and $c_1(\mathcal{F}) = -A_1\sigma_1 - A_1'\sigma_2 + \pi^*\lambda$ with $(A_1 + A_1') \geq 0$ and $\lambda H \leq (A_1 c_1 + \lambda)H < 0$ (the same holds for A_1')

$$c_1(\mathcal{F})J^2 = -(A_1 + A_1')(H - zc_1)^2 + 2z(2H - zc_1)\lambda < 0. \qquad (4.2)$$

4.2 Non-split condition

We derive a condition such that the extension can be chosen non-split (as necessary for V_4 stable, cf. (2.3)). An extension can be chosen non-split if (with $\mathcal{E} := E_1 \otimes E_2^*$)

$$Ext^1\Big(\pi^* E_2 \otimes \mathcal{O}_X(D), \pi^* E_1 \otimes \mathcal{O}_X(-D)\Big) = H^1\Big(X, \pi^*\mathcal{E} \otimes \mathcal{O}_X(-2D)\Big) \neq 0 \qquad (4.3)$$

In the following we will discuss the case with $x > 0$. One has (let $y = 2x$)

$$x > 0: \quad \pi_*\mathcal{O}_X(-y\Sigma) = 0, \quad R^1\pi_*\mathcal{O}_X(-2x\Sigma) = \mathcal{O}_B \oplus K_B \oplus 2K_B^{-1} \oplus \ldots \oplus 2K_B^{1-y} \qquad (4.4)$$

(cf. appendix). The Leray spectral sequence yields then the isomorphism

$$H^1\Big(X, \pi^*\mathcal{E} \otimes \mathcal{O}_X(-2D)\Big) \simeq H^0\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes R^1\pi_*\mathcal{O}_X(-y\Sigma)\Big) \qquad (4.5)$$

(cf. appendix). Because of (4.4) it suffices to show that one of the terms in the corresponding decomposition of (4.5) is non-vanishing. We give sufficient conditions for $H^0(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes K_B^{1-y}) \neq 0$. For this we will compute the expression

$$\chi\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes K_B^{1-y}\Big) = \sum_{i=0}^{2} (-1)^i \dim H^i \Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes K_B^{1-y}\Big) \qquad (4.6)$$

So the sequence defining V_4 can be chosen non-split if $\chi(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes K_B^{1-y}) > 0$ and $H^2(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes K_B^{1-y}) = H^0\Big(B, \mathcal{E}^* \otimes \mathcal{O}_B(2\alpha) \otimes K_B^y\Big)^* = 0$. Now $H^0(B, \mathcal{E}^* \otimes \mathcal{O}_B(2\alpha) \otimes K_B^y) = 0$ if $\mu(\mathcal{E}^* \otimes \mathcal{O}_B(2\alpha) \otimes K_B^y) = 12(2\alpha - yc_1)z(2h - z)c_1 < 0$ (a section gives a slope zero subbundle and $\mathcal{E}^*$ is semistable ([18], Thm. 10.16) of zero-slope), i.e., if $2\alpha - yc_1 = 2(\alpha - xc_1) < 0$. The Hirzebruch-Riemann-Roch theorem gives the condition

$$\begin{aligned} 0 &< \frac{1}{2}\chi\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes K_B^{1-y}\Big) \\ &= \frac{1}{2}\int_B ch(\mathcal{E})ch(\mathcal{O}_B(-2\alpha))ch(K_B^{1-y})Td(B) \\ &= 2 + \Big((y-1) + (y-1)^2\Big)c_1^2 + 4\alpha^2 - 2\Big(1 + 2(y-1)\Big)\alpha c_1 - \Big(k_1 + k_2\Big) \qquad (4.7) \end{aligned}$$

So $\alpha - xc_1 < 0$ and (4.7) are sufficient conditions for the existence of a non-split extension.

The existence of an *invariant* extension bundle V_4 follows by the same arguments as on B: the pull-back bundles $\pi^* E_i$ are τ_X-invariant as the E_i are chosen τ_B-invariant (the action in the elliptic fiber coordinates is ineffective here); the divisor Σ is invariant and $\pi^*\alpha$ can be chosen invariant as α itself can from the selection of monomials argument.

4.3 Stability of the rank 4 extension

Having shown that (4.1) can be chosen non-split for $x > 0$ we give the stability proof along the lines of [1], giving a range in the Kähler cone such that V_4 is stable. Consider

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & P = \bar{P} \otimes \mathcal{O}_X(-D) & \to & V/V'_{r+s} & \to & T = \bar{T} \otimes \mathcal{O}_X(D) & \to & 0 \\ & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & \pi^* E_1 \otimes \mathcal{O}_X(-D) & \overset{i}{\to} & V_4 & \overset{j}{\to} & \pi^* E_2 \otimes \mathcal{O}_X(D) & \to & 0 \\ & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & F_r \otimes \mathcal{O}_X(-D) & \to & V'_{r+s} & \to & G_s \otimes \mathcal{O}_X(D) & \to & 0 \\ & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad (4.8)$$

where $F_r \otimes \mathcal{O}_X(-D) = i^{-1}V'_{r+s}$ and $G_s \otimes \mathcal{O}_X(D) = j(V'_{r+s})$ are of rank $0 \leq r \leq 2$ and $0 \leq s \leq 2$. First note thar for a general subsheaf V'_{r+s} of V one has

$$(r+s)\mu(V'_{r+s}) = r\mu(F_r) + s\mu(G_s) + (s-r)DJ^2 \qquad (4.9)$$

To prove stability of V_4 we have to show that $\mu(V'_{r+s}) < 0$ for $0 \le r \le 2$ and $0 \le s \le 2$ with $0 < r + s < 4$, so the cases $(0,0)$ and $(2,2)$ do not have to be considered. The cases $(2,s)$ with $0 \le s < 2$ do not have to be considered as we can assume (cf. Lemma 4.5, [17]) that the quotient V_4/V'_{r+s} is torsion free, but the quotient $\pi^*E_1 \otimes \mathcal{O}_X(-D))/(F_2 \otimes \mathcal{O}_X(-D))$ is a torsion sheaf, thus zero, i.e. $F_2 = \pi^*E_1$ and we assume anyway the necessary condition $DJ^2 > 0$ such that $\mu(\pi^*E_1 \otimes \mathcal{O}_X(-D))) < 0$ (cf. the discussion after (2.3)). Note also that $r = 0$ implies $F_r = 0$ as π^*E_1 does not have a non-zero torsion subsheaf (same for $s = 0$).

As the pullback bundles are stable we have for $0 < r < 2$ and $0 < s < 2$ that

$$\mu(F_r) < 0, \quad \mu(G_s) < 0. \tag{4.10}$$

We also have $\mu(F_2) \le 0$ and $\mu(G_2) \le 0$ (cf. [17]). So the cases (r,s) with $(1,1)$ and $(1,0)$ are done as already $\mu(V'_{r+s}) < 0$ and we are left with the cases $(1,2)$, $(0,1)$ and $(0,2)$. As in [1] we treat these cases by solving the corresponding slope inequalities for z thus determining a range in the Kähler cone $\mathcal{C}_X$ where V_5 is stable. Note first that

$$\begin{aligned}
r\mu(F_r) &= -(A_1 + A'_1)(h - z)^2 c_1^2 + 2z(2h - z)\lambda_1 c_1 \\
s\mu(G_s) &= -(A_2 + A'_2)(h - z)^2 c_1^2 + 2z(2h - z)\lambda_2 c_1 \\
DJ^2 &= 2x(h - z)^2 c_1^2 + 2z(2h - z)\alpha c_1
\end{aligned} \tag{4.11}$$

where $(A_1 + A'_1) \ge 0$ and $(A_2 + A'_2) \ge 0$. Further we will assume that $x > 0$ and $DJ^2 > 0$.

The cases $(1,2)$ and $(0,1)$: as $\mu(G_2) \le 0$, it is sufficient to solve $\mu(F_1) + DJ^2 < 0$ for z. As $(A_1 + A'_1) \ge 0$ we will assume $(A_1 + A'_1) = 0$ as this gives the strongest condition. Similarly from the stability of π^*E_1 we have $\lambda_1 c_1 < 0$ and the condition will become strongest for $\lambda_1 c_1 = -2$. Let $\zeta := h - z$ such that $0 < \zeta < h$. Then the condition becomes

$$x\zeta^2 + (\alpha c_1 - 2)(h^2 - \zeta^2) < 0 \tag{4.12}$$

We immediately find the necessary condition $\alpha c_1 - 2 < 0$ or

$$\alpha c_1 \le 0 \tag{4.13}$$

Recall that from the non-split condition we assume $\alpha - x c_1 < 0$. Then solving the estimated inequality for z we find the bound (which becomes strongest for $\lambda c_1 = -2$)

$$h^2 - \zeta^2 > \frac{x c_1^2}{-(\alpha - x c_1)c_1 - \lambda_1 c_1} h^2 \tag{4.14}$$

A similar discussion leads to (4.14) in the $(0,1)$ case (then with λ_2).

The case $(0,2)$: here one has to solve $2\mu(G_2) + 2DJ^2 < 0$ for z. As $DJ^2 > 0$ and $\mu(G_2) \le 0$ we would have $\mu(V'_{0+2}) > 0$ if $\mu(G_2) = 0$ which would destabilize V_4 if such a case could actually occur. So we have to make sure that subsheaves V' of type $(0,2)$ with $\mu(G_2) = 0$ do not occur. This argument, involving the so-called f-map, is given below.

So let us suppose that we can assume $\mu(G_2) < 0$; so we have to solve for z

$$2\mu(G_2) + 2DJ^2 < 0 \tag{4.15}$$

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where we have to treat the cases $A_2 + A_2' = 0$ with $\lambda_2 c_1 = -2$ and $A_2 + A_2' = 1$ with $\lambda_2 c_1 = 0$ (the latter is now possible, cf. below; we assume here the minimal values of $A_2 + A_2'$ and $\lambda_2 c_1$ corresponding to the largest values of $\mu(G_2)$). For $A_2 + A_2' = 0$ and $\lambda c_1 = -2$ we get the bound (which is stronger than (4.14))

$$h^2 - \zeta^2 > \frac{2xc_1^2}{-2(\alpha - xc_1)c_1 + 2} \, h^2 \tag{4.16}$$

and for $A_2 + A_2' = 1$ and $\lambda_2 c_1 = 0$ we get (as $4(\alpha - xc_1)c_1 + c_1^2 < 0$)

$$h^2 - \zeta^2 > \frac{(4x - 1)c_1^2}{-4(\alpha - xc_1)c_1 - c_1^2} \, h^2 \tag{4.17}$$

Comparing the above bounds we find the strongest bound is (4.16) provided that $\alpha c_1 < 1 - 4x$ (otherwise it is (4.17)).

In order that (4.16) can be solved the ratio must be less than 1, which again expresses the condition (4.13). The condition $DJ^2 > 0$ imposes the upper bound

$$h^2 - \zeta^2 < \frac{xc_1^2}{-(\alpha - xc_1)c_1} \, h^2 \tag{4.18}$$

thus we find in total the condition (provided that $\alpha c_1 < 1 - 4x$, cf. above)

$$\frac{xc_1^2}{-(\alpha - xc_1)c_1 + 1} \, h^2 < h^2 - \zeta^2 < \frac{xc_1^2}{-(\alpha - xc_1)c_1} \, h^2 \tag{4.19}$$

So, under the mentioned assumptions, V_4 is stable with respect to $J = z\Sigma + h\pi^* c_1$ if (4.19) is satisfied for $\zeta = h - z$.

The f-map argument. We still have to make sure that subsheaves of type $(0, 2)$ with $\mu(G_2) = 0$ do not occur. More generally, one can pose a condition such that a subsheaf of type $(0, 2)$ does not exist.

Let us recall the general argument from [1]. Let $U := \pi^* E_1 \otimes \mathcal{O}_X(-D)$, $W := \pi^* E_2 \otimes \mathcal{O}_X(D)$ and $G := G_2 \otimes \mathcal{O}_X(D)$. A sufficient condition for V_4 not to be destabilized by a subsheaf G of W is given by injectivity of the map (which we will call the *f-map*)

$$\mathrm{Ext}^1(W, U) \xrightarrow{f} \mathrm{Ext}^1(G, U) \tag{4.20}$$

To see this we ask when is it possible that a map $G \to W$ lifts to a map $G \to V$. Consider

$$\to \mathrm{Hom}(G, V) \to \mathrm{Hom}(G, W) \to \mathrm{Ext}^1(G, U) \tag{4.21}$$

showing that the obstruction to lifting an element of $\mathrm{Hom}(G, W)$ to an element of $\mathrm{Hom}(G, V)$ lies in $\mathrm{Ext}^1(G, U)$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(W, W) & \xrightarrow{\partial} & \mathrm{Ext}^1(W, U) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(G, W) & \to & \mathrm{Ext}^1(G, U) \end{array} \tag{4.22}$$

with $\partial(1) = \xi$ the extensions class. So we conclude a non-zero element of $\mathrm{Hom}(G, W)$ can be lifted to an element of $\mathrm{Hom}(G, V)$ exactly when the extension class ξ is in the kernel of

$$f \colon \mathrm{Ext}^1(W, U) \to \mathrm{Ext}^1(G, U) \tag{4.23}$$

Thus if f is injective $f(\xi) \neq 0$ and such a lifting does not exist.

Before specifying the *f-map* in our situation let us determine $c_1(G_2)$ and show that $\mu(G_2) = 0$ if and only if $c_1(G_2) = 0$. From $G_2 \to \pi^* E_2$, we have $(\Lambda^2 G_2)^{**} \to \Lambda^2 \pi^* E_2^{**}$ now $rk(\Lambda^2 G_2) = 1$ and $(\Lambda^2 G_2)^{**}$ is a reflexive torsion free sheaf of rank one (i.e., a line bundle); furthermore $c_1(G_2) = c_1((\Lambda^2 G_2)^{**})$. As $(\Lambda^2 \pi^* E_2)^{**} =: L$ with $c_1(L) = 0$ we find

$$c_1(G_2) = -D_2 \tag{4.24}$$

with D_2 an effective divisor (set $D_2 = A_2 \sigma_1 + A_2' \sigma_2 - \pi^* \lambda_2$). The slope of G_2 is then given by (4.11) and as $\lambda_2 c_1 \leq 0$ and $A_2 + A_2' \geq 0$ we get $\mu(G_2) = 0$ if and only if $c_1(G_2) = 0$.

Let us specify the *f-map* in our case. Consider the exact sequence

$$0 \to G_2(D) \to \pi^* E_2(D) \to T(D) \to 0 \tag{4.25}$$

where $T = \pi^* E_2 / G_2$ is a torsion sheaf. Applying $Hom(\cdot, \pi^* E_1 \otimes \mathcal{O}_X(-2D))$ we get

$$Ext^1(T(D), \pi^* E_1(-2D)) \to Ext^1(\pi^* E_2(D), \pi^* E_1(-D)) \xrightarrow{f} Ext^1(G_2(D), \pi^* E_1(-D)) \tag{4.26}$$

Thus we have to show that $Ext^1(T(D), \pi^* E_1(-D)) = 0$. We have

$$Ext^1(T(D), \pi^* E_1(-D)) = Ext^2(\pi^* E_1(-D), T(D))^* = H^2(X, \pi^* E_1^*(2D) \otimes T)^* \tag{4.27}$$

We know $\mu(G_2) = 0$ if and only if $c_1(G_2) = 0$. As T is a torsion sheaf we have $T = j_* T'$ with $j \colon Y \to X$ and T' some sheaf on Y and the Grothendieck-Riemann-Roch theorem gives $0 = c_1(T) = c_1(j_* T') = j_*(rk(T')) = rk(T') Y_{co\,1}$. Thus T is supported in codimension ≥ 2 and H^2 vanishes. So destabilizing subsheaves of type $(0, 2)$ with $\mu(G_2) = 0$ do not occur.

4.4 The Bogomolov inequality

As a check on our proof of stability we will derive that the Bogomolov inequality

$$c_2(V_4) J \geq 0 \tag{4.28}$$

for a stable bundle with $c_1(V_4) = 0$ is fulfilled. This becomes in our case

$$\begin{aligned}
\frac{1}{2} c_2(V_4) J &= \frac{1}{2} \Big((k_1 + k_2) F - 2 \big[x(2\alpha - xc_1)\Sigma + \alpha^2 \big] \Big)(z\Sigma + hc_1) \\
&= -(h - z) 2x \Big((\alpha - xc_1)c_1 + \alpha c_1 \Big) - 2z\alpha^2 + z(k_1 + k_2) \\
&\geq 0
\end{aligned} \tag{4.29}$$

We have $x > 0$, $h - z > 0$, $\alpha c_1 \leq 0$ and $\alpha - xc_1 < 0$ showing that $c_2(V_4)J \geq 0$ *if one is in the case $\alpha^2 \leq 0$*. In the general case (4.18) becomes (with the notation $\beta := -\alpha$)

$$h^2 - (h-z)^2 < \frac{xc_1^2}{xc_1^2 + \beta c_1} \, h^2$$

$$\Longleftrightarrow \qquad h^2 \beta c_1 < (h-z)^2(xc_1^2 + \beta c_1)$$

$$\Longleftrightarrow \qquad \frac{h}{h-z} < \left(\frac{h}{h-z}\right)^2 < \frac{xc_1^2 + \beta c_1}{\beta c_1} \leq \frac{x^2c_1^2 + 2x\beta c_1 + \beta^2}{\beta^2} \qquad (4.30)$$

using the inequality (note that c_1 is here ample)

$$\beta^2 c_1^2 \leq (\beta c_1)^2 \qquad (4.31)$$

and $\beta c_1 \geq 0$ (actually we can even assume $\beta c_1 > 0$ as otherwise $\alpha^2 \leq 0$ by the Hodge index theorem when (4.29) was clear). So one has indeed (note $k_i \geq 0$)

$$h\beta^2 < (h-z)(x^2c_1^2 + 2x\beta c_1 + \beta^2) \quad \Longleftrightarrow \quad 0 < (h-z)x(xc_1^2 + 2\beta c_1) - z\beta^2 \quad (4.32)$$

5. Physical constraints

5.1 Breaking the $SU(5)$ GUT group to the standard model gauge group

On X' one turns on a $\mathbf{Z_2}$ Wilson line of generator $\mathbf{1_3} \oplus -\mathbf{1_2}$ breaking $H = SU(5)$ to H_{SM}

$$SU(5) \longrightarrow H_{SM} = SU(3)_c \times SU(2)_{ew} \times U(1)_Y \qquad (5.1)$$

(up to a $\mathbf{Z_6}$). This gives, from $\bar{\mathbf{5}} = \bar{d} \oplus L$ and $\mathbf{10} = Q \oplus \bar{u} \oplus \bar{e}$, the fermionic matter content

$$SM \; fermions = Q \oplus L \oplus \bar{u} \oplus \bar{d} \oplus \bar{e}$$

$$= (\mathbf{3}, \mathbf{2})_{1/3} \oplus (\mathbf{1}, \mathbf{2})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-4/3} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{2/3} \oplus (\mathbf{1}, \mathbf{1})_2 \qquad (5.2)$$

of the Standard model. ad_{E_8} decomposes under $G \times H = SU(5)_{str.gr.} \times SU(5)_{gau.gr.}$

$$\mathbf{248} = (\mathbf{5}, \mathbf{10}) \oplus (\bar{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\mathbf{10}, \bar{\mathbf{5}}) \oplus (\overline{\mathbf{10}}, \mathbf{5}) \oplus (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}) \qquad (5.3)$$

For $SU(5)$ GUT models with matter $\bar{\mathbf{5}} \oplus \mathbf{10}$ in one family one needs to consider besides the fundamental $V = \mathbf{5}$ to get the $\mathbf{10}$-matter also the $\Lambda^2 V = \mathbf{10}$ to get the $\bar{\mathbf{5}}$-matter; as the $\mathbf{10}$ and the $\bar{\mathbf{5}}$ come in the same number of families (as also demanded by anomaly considerations) it is enough to adjust $\chi(X, V)$ to get all the Standard model fermions.

All the above considerations concern the net generation number, i.e., the number of generations minus the number of anti-generations. Beyond the mentioned multiplets ideally a string model should provide no further exotic matter multiplets of nonzero net generation number (conjugate pairs should pair up and become massive at the string scale). In a model resembling the MSSM one furthermore wants to have just one conjugate pair $H, \bar{H}$ of Higgs doublets. Their pairing is described field-theoretically by the $\mu - term$ $\mu H \bar{H}$ where one has to understand that μ sits at the electro-weak scale and not at the string scale, say, when coming from a string model; in that case the coupling is field-dependent and mediated by a superpotential term $\lambda \phi H \bar{H}$ where λ is the coupling constant and ϕ a superfield which is, just like the right-handed natutrino ν_R, a singlet under the Standard Model gauge group, for example a modulus. If the latter aquires a vev (there may be an additional superpotential coupling $\mathbf{1}^3$ for ϕ) it provides an effective μ-term.

5.2 Building the rank 5 bundle from the rank 4 bundle

We explain now more precisely how to embed the structure group into E_8 and to get an SU(5) GUT group (cf. also [19, 20]). First we twist with a line bundle $\mathcal{O}(-\pi^*\beta)$ and build a split extension (direct sum) to embed the resulting SU(4) $\otimes$ U(1) bundle into E_8

$$0 \to V_4 \otimes \mathcal{O}_X(-\pi^*\beta) \to V_5 \to \mathcal{O}_X(4\pi^*\beta) \to 0 \tag{5.4}$$

The bundle V_5 has the Chern classes

$$
\begin{aligned}
c_2(V_5) &= -2x(2\alpha - xc_1)\Sigma - 2\alpha^2 - 10\beta^2 + k_1 + k_2 \\
\frac{c_3(V_5)}{2} &= 2x\Big[k_1 - k_2 - 4\alpha\beta + 2x\beta c_1\Big]
\end{aligned}
\tag{5.5}
$$

Note that V_5, as a direct sum, can not be stable, but only poly-stable, (direct sum of stable bundles of the same slope). This common slope must be zero as, quite generally, for a rank n bundle $\mathcal{V} = \oplus \mathcal{V}_i$ composed of U(n_i) bundles of slopes $\mu_i = \frac{1}{n_i} \int J^2 c_1(\mathcal{V}_i) =: \mu$ (they must coincide for $\mathcal{V}$ to be polystable) one finds $\int J^2 c_1(\mathcal{V}_i) = 0$ for all i as $0 = \int J^2 c_1(\mathcal{V}) = \sum r_i \mu_i = \mu n$. For us, having $\mu_i = \beta J^2 = 2z(2h - z)\beta c_1$, this means

$$\beta\, c_1 = 0 \tag{5.6}$$

More generally one gets the condition $\pi^*\beta \cdot J^2 = 0$. Its possible violation may be understood either as pointing to the necessary inclusion of one-loop effects so that the DUY condition is fulfilled quantum-mechanically [21, 7, 22]; alternatively the resulting instability can be interpreted [22] as pointing to a dynamical vacuum shift, i.e., indicating that the stable vacuum to be considered is actually a *non-split* extension (5.4).

5.3 U(n) bundles and line bundle twists of SU(n) bundles

From an SU(n) bundle V and a line bundle $\mathcal{O}_X(-\pi^*\beta)$ one can build a U(n) bundle by the twist

$$\mathcal{V} = V \otimes \mathcal{O}_X(-\pi^*\beta) \tag{5.7}$$

So $c_1(\mathcal{V}) \equiv 0\,(n)$ as $\pi^*\beta$ is integral. Conversely, if $c_1(\mathcal{V}) \equiv 0\,(n)$, one can split off an integral class $\pi^*\beta$ of $c_1(\mathcal{V}) = n\pi^*\beta$ and define a corresponding line bundle $\mathcal{O}_X(-\pi^*\beta)$ such that $V := \mathcal{V} \otimes \mathcal{O}_X(\pi^*\beta)$ is an SU(n) bundle, i.e., one can think of $\mathcal{V}$ then as $V \otimes \mathcal{O}_X(-\pi^*\beta)$.

Note that the structure group U(n) arises in the decomposing case from SU(n) $\cdot$ U(1) (the latter factor is understood here always as embedded by multiples of the identity matrix) whereas for a bundle $V \oplus \mathcal{L}(D)$ the structure group would be the direct product SU(n) $\times$ U(1). Note that there is a morphism $f\colon \mathrm{SU}(n) \times \mathrm{U}(1) \to \mathrm{U}(n)$ sending $(a, b) \mapsto a \cdot b$. The image of this morphism is U(n) = SU(n) $\cdot$ U(1), so SU(n) $\cdot$ U(1) = $\Big(\mathrm{SU}(n) \times \mathrm{U}(1)\Big)/\ker(f)$. The subgroup $\ker(f)$ is formed by all pairs $(\lambda \cdot \mathrm{Id}_n, \lambda^{-1})$ where $\lambda \in \mathbf{C}$ with $\lambda \cdot \mathrm{Id}_n \in \mathrm{SU}(n)$, i.e., $\lambda^n = 1$ and $\ker(f) = \mathbf{Z_n}$ (the group of n-th roots of unity). As the difference between the direct product and the product is just a discrete group, and since all group theoretical statements in this paper are understood on the level of Lie-algebras, we will write SU(4) $\times$ U(1) instead of SU(4) $\cdot$ U(1) for our structure group G.

5.4 E_8 embedding and massive U(1)

Let us make the embedding of G in E_8 more explicit. One embeds a U(4) bundle block-diagonally via

$$\mathrm{U}(4) \ni A \to \begin{pmatrix} A & 0 \\ 0 & \det^{-1} A \end{pmatrix} \in \mathrm{SU}(5) \tag{5.8}$$

Therefore, after making the twist $\mathcal{V} = V_4 \otimes \mathcal{O}_X(-\pi^*\beta)$ with $c_1(\mathcal{V}) = -4\pi^*\beta$ one actually has to work with the bundle

$$V_5 = \mathcal{V} \oplus \mathcal{O}_X(4\pi^*\beta) = V_4 \otimes \mathcal{O}_X(-\pi^*\beta) \oplus \mathcal{O}_X(4\pi^*\beta) \tag{5.9}$$

The unbroken gauge group will then be given by $H = \mathrm{SU}(5) \times \mathrm{U}(1)_X$, the commutator of $G = \mathrm{SU}(4) \times \mathrm{U}(1)_X$ in the observable E_8. The decomposition $ad(E_8) = \bigoplus_i U_i^{\mathrm{SU}(4)} \otimes R_i^{\mathrm{SO}(10)} = \bigoplus_i (U_i, R_i) = \bigoplus_i (U_i, S_i^{\mathrm{SU}(5)})_{t_i^{\mathrm{U}(1)}}$ of the adjoint representation of E_8 under $\mathrm{SU}(4) \times \mathrm{SU}(5) \times \mathrm{U}(1)_X$ specifies itself as follows

$$\begin{aligned} \mathbf{248} \longrightarrow\ & (\mathbf{5}, \mathbf{10}) \oplus (\bar{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\mathbf{10}, \bar{\mathbf{5}}) \oplus (\overline{\mathbf{10}}, \mathbf{5}) \oplus (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}) \\ \longrightarrow\ & \Big((\mathbf{4}, \mathbf{1})_{-5} \oplus (\mathbf{4}, \bar{\mathbf{5}})_3 \oplus (\mathbf{4}, \mathbf{10})_{-1} \Big) \oplus \Big((\bar{\mathbf{4}}, \mathbf{1})_5 \oplus (\bar{\mathbf{4}}, \mathbf{5})_{-3} \oplus (\bar{\mathbf{4}}, \overline{\mathbf{10}})_1 \Big) \\ & \oplus (\mathbf{6}, \mathbf{5})_2 \oplus (\mathbf{6}, \bar{\mathbf{5}})_{-2} \oplus (\mathbf{15}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{10})_4 \oplus (\mathbf{1}, \overline{\mathbf{10}})_{-4} \oplus (\mathbf{1}, \mathbf{24})_0 \end{aligned} \tag{5.10}$$

The SU(5) representations are given as an auxiliary step. The full deomposition, identical to an auxiliary $\mathrm{SU}(4) \times \mathrm{SO}(10)$ step, leads to the right-handed neutrino ν_R; additionally (besides the gauge bosons $(\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{24})_0$ of H) some neutral matter given by singlets (moduli) arises from $End(V)$, i.e., $(\mathbf{15}, \mathbf{1})_0$. The massless (charged) matter content is

$$\bigoplus (S_k)_{t_k} = \mathbf{1}_{-5} \oplus \bar{\mathbf{5}}_3 \oplus \mathbf{10}_{-1} \oplus \bar{\mathbf{5}}_{-2} \oplus \mathbf{10}_4 \tag{5.11}$$

The first three multiplets refer to the ν_R plus the SM fields. The $\bar{\mathbf{5}}_{-2}$ refers to Higgses related to $\Lambda^2 V$; the last multiplet $\mathbf{10}_4$ describes further exotic matter. Ideally one would want, of course, to avoid net generations of the last two multiplets.

Precisely those U(1)'s in the gauge group H which occur already in the structure group G (so-called U(1)'s of type I, other U(1)'s in H are called to be of type II) are anomalous [21]–[23 – 25]. The anomalous $\mathrm{U}(1)_X$ can gain a mass by absorbing some of the would be massless axions via the Green-Schwarz mechanism, that is, the gauge field is eliminated from the low energy spectrum by combining with an axion and so becoming massive. One has to check that the anomalies related to $\mathrm{U}(1)_X$ do not cancel accidentally (i.e., that the mixed abelian-gravitational, the mixed abelian-non-abelian and the pure cubic abelian anomaly do not all vanish). Computing the anomaly-coefficients of $\mathrm{U}(1)_X$ one finds for (5.9)

$$\begin{aligned} A_{\mathrm{U}(1)-G_{\mu\nu}^2} &= \sum tr_{(S_k)_{t_k}} q \cdot \chi(X, U_k \otimes t_k) &= 10\beta \cdot \Big(12c_2(V) - 5c_2(X) \Big) \\ A_{\mathrm{U}(1)-\mathrm{SU}(5)^2} &= \sum q_{t_k} C_2(S_k) \cdot \chi(X, U_k \otimes t_k) &= 10\beta \cdot \Big(2c_2(V) - c_2(X) \Big) \\ A_{\mathrm{U}(1)^3} &= \sum tr_{(S_k)_{t_k}} q^3 \cdot \chi(X, U_k \otimes t_k) &= 600\beta \cdot \Big(2c_2(V) - c_2(X) \Big) \end{aligned} \tag{5.12}$$

(with C_2 normalised to give $C_2(\bar{f}) = 1$, $C_2(\Lambda^2 f) = 3$ for SU(5)). The first condition is generically independent, so not all three coefficients would vanish. Note that in our specific case, after taking into account the condition $\beta c_1 = 0$, one has with

$$\beta c_2(V) = \beta \cdot (-4x)(2\alpha - xc_1) = -8x\alpha\beta \quad, \qquad \beta c_2(X) = 0 \tag{5.13}$$

that all three expressions will be proportional to $\alpha\beta$. So we get for the decoupling of the additional U(1) in the gauge group the condition

$$\alpha\,\beta \neq 0 \tag{5.14}$$

5.5 Avoiding exotic matter

Recall the massless (charged) matter content

$$\bigoplus (S_k)_{t_k} = \mathbf{1_{-5}} \oplus \mathbf{\bar{5}_3} \oplus \mathbf{10_{-1}} \oplus \mathbf{\bar{5}_{-2}} \oplus \mathbf{10_4} \tag{5.15}$$

Here we have a Standard Model fermion generation $\mathbf{\bar{5}_3} \oplus \mathbf{10_{-1}}$ plus a right-handed neutrino $\mathbf{1_{-5}}$. Besides these multiplets the $\mathbf{\bar{5}_{-2}}$ refers to Higgses related to $\Lambda^2 V$; furthermore we have exotic matter $\mathbf{10_4}$ which we would like to avoid. The net-amount of such states (which could not pair up and become massive) is computed from $\chi(X, U_k \otimes t_k)$

matter multiplet	net-amount
$(\mathbf{4}, \mathbf{10})_{-1}$	$\chi(X, V \otimes \mathcal{L}^{-1})$
$(\mathbf{1}, \mathbf{10})_4$	$\chi(X, \mathcal{L}^4)$
$(\mathbf{4}, \mathbf{\bar{5}})_3$	$\chi(X, V \otimes \mathcal{L}^3)$
$(\mathbf{6}, \mathbf{\bar{5}})_{-2}$	$\chi(X, \Lambda^2 V \otimes \mathcal{L}^{-2})$
$(\mathbf{4}, \mathbf{1})_{-5}$	$\chi(X, V \otimes \mathcal{L}^{-5})$

One gets for the individual terms with $\beta c_1 = 0$

$$
\begin{aligned}
\chi(V \otimes \mathcal{L}^{-1}) &= \frac{c_3(V)}{2} - \beta\Big(\frac{c_2(X)}{3} - c_2(V)\Big) \\
&= 2x\Big(k_1 - k_2 - 4\alpha\beta\Big) \\
\chi(\mathcal{L}^4) &= \beta\frac{c_2(X)}{3} \\
&= 0 \\
\chi(V \otimes \mathcal{L}^3) &= \frac{c_3(V)}{2} + \beta\Big(c_2(X) - 3c_2(V)\Big) \\
&= 2x\Big(k_1 - k_2 + 12\alpha\beta\Big) \\
\chi(\Lambda^2 V \otimes \mathcal{L}^{-2}) &= -\beta\Big(c_2(X) - 4c_2(V)\Big) \\
&= 2x\Big(-16\alpha\beta\Big) \\
\chi(V \otimes \mathcal{L}^{-5}) &= \frac{c_3(V)}{2} - 5\beta\Big(\frac{c_2(X)}{3} - c_2(V)\Big) \\
&= 2x\Big(k_1 - k_2 - 20\alpha\beta\Big)
\end{aligned}
\tag{5.16}
$$

Note as a check that the non-abelian anomaly vanishes

$$\chi(V \otimes \mathcal{L}^{-1}) + \chi(\mathcal{L}^4) - \chi(V \otimes \mathcal{L}^3) - \chi(\Lambda^2 V \otimes \mathcal{L}^{-2}) = 0 \tag{5.17}$$

Equivalently the net-chirality $\chi(V \otimes \mathcal{L}^3) + \chi(\Lambda^2 V \otimes \mathcal{L}^{-2})$ of the $\mathbf{\bar{5}}$-matter and $\chi(V \otimes \mathcal{L}^{-1}) + \chi(\mathcal{L}^4)$ of the $\mathbf{10}$-matter coincide as they should. Avoiding a non-zero net chirality from the Higgs sector (also there would be no mass terms for unpaired $\mathbf{10}_4$'s) we may demand

$$N_{\text{gen}}(\Lambda^2 V \otimes \mathcal{L}^{-2}) = 0 = N_{\text{gen}}(\mathcal{L}^4) \tag{5.18}$$

Interpreting these generation numbers as indices gives the vanishing conditions $N_{\text{gen}}(\mathcal{L}^4) = 0$ (which means $\beta c_2(X) = 0$ and is automatically fulfilled for $\beta c_1 = 0$) and furthermore $N_{\text{gen}}(\Lambda^2 V \otimes \mathcal{L}^{-2}) = -32x\alpha\beta = 0$ (which in view of the previous condition means $\beta c_2(V) = 0$). Furthermore one has then just $N_{\text{gen}} = \chi(V \otimes \mathcal{L}^{-1}) = c_3(V)/2$.

One gets from $N_{\text{gen}}(\Lambda^2 V \otimes \mathcal{L}^{-2}) = 0$ the condition

$$\alpha\beta = 0 \tag{5.19}$$

Therefore one has to face the following alternative: either one proceeds as described in (5.19) and avoids the exotic matter, then the U(1) in the gauge group has not decoupled; or, alternatively, one insists on this decoupling, i.e. on the condition (5.14) that the anomaly is present, but keeps then the exotic matter. (An extended analysis shows that the situation is not improved by considering $D' = y\Sigma + \pi^*\beta$ with $y \neq 0$.)

6. List of constraints and solutions

Let us list all the constraints we have found. Keep in mind that we assume always $x > 0$ and $k_1, k_2 \geq 0$; furthermore we implement (5.6) (note also (5.5))

$$
\begin{aligned}
\alpha - xc_1 &< 0 \\
0 < \tfrac{1}{2}\chi &= 2 + 2c_1^2 - 6\alpha c_1 + 4\alpha^2 - (k_1 + k_2) \qquad \text{for} \quad x = 1 \\
\alpha c_1 &\leq 0 \\
w_B = (6 - 2x^2)c_1 + 4x\alpha &\geq 0 \qquad\qquad\qquad \xrightarrow{x=1} \ \alpha \geq -c_1 \\
a_f = 44 + 2\alpha^2 + 10\beta^2 - (k_1 + k_2) &\geq 0 \\
\tfrac{1}{2}N_{\text{gen}} = x\Big(k_1 - k_2 - 4\alpha\beta\Big) &= \pm 3
\end{aligned}
\tag{6.1}
$$

From $w_B \geq 0$ together with $\alpha c_1 \leq 0$ (such that $\alpha = (p,q)$ can not have $p,q > 0$) one finds

$$x = 1 \tag{6.2}$$

In writing N_{gen} we have already used the slope condition $\beta c_1 = 0$ from (5.6). Therefore, on $\mathbf{F_0} = \mathbf{P^1} \times \mathbf{P^1}$ one gets explicitly that $\beta = (e, -e)$, or more precisely with $a_f \geq 0$

$$\beta = \pm(1, -1) \tag{6.3}$$

(using that $\alpha^2 \leq 4$). Furthermore, one has the following reflection property

$$\Big(\alpha = (p,q), \ (k_1,k_2) \longrightarrow N_{\text{gen}} = \pm 6\Big) \Longrightarrow \Big(\alpha = (q,p), \ (k_2,k_1) \longrightarrow N_{\text{gen}} = \mp 6\Big) \quad (6.4)$$

One has to distinguish two cases. One can implement either the condition $\alpha\beta = 0$ avoiding so exotic matter but keeping the additional $\text{U}(1)$ in the low-energy spectrum. This is the case we are going to describe now. Alternatively, one can choose the inverted option (keeping exotic matter but avoiding the additional $\text{U}(1)$) if $\alpha\beta \neq 0$; this is discussed further in the appendix D.

So let us study here the case $\alpha\beta = 0$ where we avoid the exotic matter (but keep an additional $\text{U}(1)$). By the conditions given above this means $\alpha = (p,p)$ with $p = -2, -1$ or 0. The latter case does not actually occur when taking into account the bounds $k_i \geq 8$: for $\alpha = (0,0)$ one would get $k_1 + k_2 < 18$ from the non-split condition $\chi > 0$ which is not solvable together with $k_1 - k_2 = \pm 3$ and $k_i \geq 8$. The latter condition is forced on us (for $h = 1/2$) as we want to use the concrete bundles E_i on the base B, constructed in appendix B.

The reflection property (6.4) implies that for each solution pair (k_1, k_2) also the further pair (k_2, k_1) occurs (by which the following list has to be augmented).

α	k_1	k_2	i
$(-2,-2)$	$8+i$	$11+i$	$i = 0, \ldots, 10$
$(-1,-1)$	$8+i$	$11+i$	$i = 0, \ldots, 4$

Here the relative size $k_1 - k_2$ of the instanton numbers is given by the physical generation number $N_{\text{gen}}^{\text{phys}} = N_{\text{gen}}/2$ (downstairs on $X' = X/\mathbf{Z_2}$). The lower bound on i comes from the conrete construction of the E_i (cf. appendix) and the upper bound stems from the condition $a_f \geq 0$ (which is stronger than the non-split condition $\chi > 0$), $k_1 + k_2 \leq 24 + 2\alpha^2$.

7. Conclusions

We build heterotic sting models with the gauge group of the Standard Model times an additional $\text{U}(1)$. The net amount of chiral matter is given by precisely three net generations of chiral fermions of the Standard Model (including a right-handed neutrino ν_R). This is done by building first the elliptic Calabi-Yau space X over $\mathbf{F_0} = \mathbf{P^1} \times \mathbf{P^1}$, the B-fibration with two sections σ_1 and σ_2 possessing a free involution τ_X leaving the holorphic three-form invariant. Then we construct an invariant $\text{SU}(5)$ model of six net generations over X which descends to the quotient Calabi-Yau $X' = X/\mathbf{Z_2}$ and is afterwards broken to the Standard Model by turning on a Wilson line; the latter is possible as $\pi_1(X') = \mathbf{Z_2}$.

The invariant $\text{SU}(5)$ model on the cover space X arises actually not from an invariant $\text{SU}(5)$ bundle but from an $\text{SU}(4) \times \text{U}(1)$ bundle (where $\beta = \pm(1,-1)$)

$$V_5 = V_4 \otimes \mathcal{O}_X(-\beta) \oplus \mathcal{O}_X(4\beta)$$

The invariant rank 4 bundle V_4 of $c_1(V_4) = 0$ is defined by a non-split extension

$$0 \rightarrow \pi^* E_1 \otimes \mathcal{O}_X(-D) \rightarrow V \rightarrow \pi^* E_2 \otimes \mathcal{O}_X(D) \rightarrow 0$$

Here D is the divisor $D = \Sigma + \pi^*\alpha$, chosen invariant, where α is one of the elements $(-2, -2)$ or $(-1, -1)$, and the E_i are two stable vector bundles on the base B of rank two, $c_1(E_i) = 0$ and $c_2(E_i) = k_i$; concretely they can be described by the construction of the appendix for $k_i \geq 8$. The list of applicable instanton numbers (k_1, k_2) is given in table 1. The physical generation number $N_{\text{gen}}^{\text{phys}} = N_{\text{gen}}/2$ (downstairs on $X' = X/\mathbf{Z_2}$) is given just by the mismatch of the instanton numbers

$$N_{\text{gen}}^{\text{phys}} = k_1 - k_2 \tag{7.1}$$

These data-sets fulfil all necessary conditions for the existence of a non-split extension, stability, DUY-equation, fivebrane effectivity and generation number.

The information about the matter content concern here just the fermions, and further only their net generation number. Extensions of these investigations describing cases with the appropriate Higgs content and the individual numbers of generations and anti-generations will be reported elsewhere.

Acknowledgments

A. Cohomological relations

Relations from the Leray spectral sequence. The Leray spectral sequence of the fibration $\pi : X \to B$ relates the cohomology of a bundle V on X to the cohomology of the higher direct image sheaves $R^i\pi_* V$ on the base B. The latter are defined by $R^i\pi_* V(\mathcal{U}) = H^1(\pi^{-1}(\mathcal{U}), V|_{\pi^{-1}(\mathcal{U})})$ for an open set $\mathcal{U} \subset B$; moreover, for any point $b \in B$ one has $R^i\pi_* V|_b = H^i(F_b, V|_{F_b})$. Applied to our situation in section 4.2 the Leray spectral sequence degenerates to the long exact sequence

$$\begin{aligned}
0 \to \, & H^1\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes \pi_* \mathcal{O}_X(-2x\Sigma)\Big) \\
\to \, & H^1\Big(X, \pi^*\mathcal{E} \otimes \mathcal{O}_X(-2D)\Big) \\
\to \, & H^0\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes R^1\pi_* \mathcal{O}_X(-2x\Sigma)\Big) \\
\to \, & H^2\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes \pi_* \mathcal{O}_X(-2x\Sigma)\Big) \\
\to \, & H^2\Big(X, \pi^*\mathcal{E} \otimes \mathcal{O}_X(-2D)\Big) \\
\to \, & H^1\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes R^1\pi_* \mathcal{O}_X(-2x\Sigma)\Big) \\
\to \, & 0
\end{aligned} \tag{A.1}$$

(using the projection formula $R^i\pi_*(V \otimes \pi^*M) = R^i\pi_*(V) \otimes M$) together with

$$\begin{aligned}
H^0\Big(X, \pi^*\mathcal{E} \otimes \mathcal{O}_X(-2D)\Big) &= H^0\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes \pi_* \mathcal{O}_X(-2x\Sigma)\Big) \\
H^3\Big(X, \pi^*\mathcal{E} \otimes \mathcal{O}_X(-2D)\Big) &= H^2\Big(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha) \otimes R^1\pi_* \mathcal{O}_X(-2x\Sigma)\Big)
\end{aligned} \tag{A.2}$$

Splitting relations. Let us prove the following relations

$$\pi_* \mathcal{O}_X(y\Sigma) = \mathcal{O}_B \oplus K_B \oplus 2K_B^2 \oplus \ldots \oplus 2K_B^y, \quad y > 1$$
$$R^1\pi_* \mathcal{O}_X(-y\Sigma) = \mathcal{O}_B \oplus K_B \oplus 2K_B^{-1} \oplus \ldots \oplus 2K_B^{1-y}, \quad y > 1 \tag{A.3}$$

We start from the exact sequence

$$0 \to \mathcal{O}_X(\sigma_1) \to \mathcal{O}_X(\Sigma) \to \pi^* K_B|_{\sigma_2} \to 0 \tag{A.4}$$

if we apply $R^i\pi_*$ and recall $R^1\pi_*\mathcal{O}_X(\sigma_1) = 0$ as well as $\pi_*\mathcal{O}_X(\sigma_1) = \mathcal{O}_B$ we find

$$0 \to \mathcal{O}_B \to \pi_*\mathcal{O}_X(\Sigma) \to K_B \to 0 \tag{A.5}$$

which splits (as $Ext^1(K_B, \mathcal{O}_B) = H^1(B, K_B^*) = 0$) and so we get

$$\pi_*\mathcal{O}_X(\Sigma) = \mathcal{O}_B \oplus K_B \tag{A.6}$$

The rest of the above formula follows by induction using ($y > 1$)

$$0 \to \pi_*\mathcal{O}_X((y-1)\Sigma) \to \pi_*\mathcal{O}_X(y\Sigma) \to \pi_*\mathcal{O}_X(y\Sigma)|_\Sigma \to 0 \tag{A.7}$$

and that $\pi_*\mathcal{O}_X(y\Sigma)|_\Sigma = 2K_B^y$ (note that σ_1 and σ_2 are disjoint). The relation for $R^1\pi_*\mathcal{O}_X(-y\Sigma)$ follows from relative duality and using the fact that the right hand side is locally free such that we can take the double dual of the left hand side[1]

$$[R^1\pi_*\mathcal{O}_X(-y\Sigma)]^* = \pi_*\mathcal{O}_X(y\Sigma) \otimes K_B^* \tag{A.8}$$

Remark: let us close with the following remark concerning the case with $x > 0$. One has

$$x > 0: \qquad \pi_*\mathcal{O}_X(-y\Sigma) = 0, \quad R^1\pi_*\mathcal{O}_X(-y\Sigma) \neq 0 \tag{A.9}$$

as $\pi_*\mathcal{O}_X(-y\Sigma)|_b = H^0(F_b, \mathcal{O}_{F_b}(-2y[p])) = 0$ since $-2y[p]$ is the negative of a non-zero effective divisor; as the degree of that divisor is negative one has $H^1(F_b, \mathcal{O}_{F_b}(-2y[p])) \neq 0$ by the Riemann-Roch theorem (or by $H^1(F_b, \mathcal{O}_{F_b}(-2y[p])) = H^0(F_b, \mathcal{O}_{F_b}(2y[p])) \neq 0$ as $2y[p]$ is a non-zero effective divisor).

B. Stable bundles on the base

To construct explicitly an appropriate rank r vector bundle V on B we consider a suitable twisted extension of $\oplus_{i=1}^{r-1} I_{Z_i}$ by $\mathcal{O}_B$. Here the I_{Z_i} are ideal sheaves of point sets Z_i. The bundle V should be stable with respect to a certain ample divisor H and H enters already the construction in form of the twist bundle. So consider the extension

$$0 \to \mathcal{O}_B(-(r-1)H) \to V \to \mathcal{O}_B(H) \otimes \bigoplus_{i=1}^{r-1} I_{Z_i} \to 0 \tag{B.1}$$

---

[1]note that since for every point $p \in B$ we have $h^1(F_p, \mathcal{O}_{F_p}(-\sigma_1(p))) = 1$, the function $p \mapsto h^1(F_p, \mathcal{O}_{F_p}(-\sigma_1(p)))$ is constant and then $R^1\pi_*\mathcal{O}_X(-\sigma_1)$ is locally free of rank 1 (cf. [26], Cor. III.12.9); the same reasoning applies for $R^1\pi_*\mathcal{O}_X(-y\Sigma)$

V is known [27] to be an H-stable bundle of $c_1(V) = 0$ and

$$c_2(V) = l(Z) - \frac{r(r-1)}{2}H^2 \tag{B.2}$$

if $l(Z) = \sum_{i=1}^{r-1} l(Z_i)$ fulfills the bound below. For $i = 1, \ldots, (r-1)$ one can choose a reduced 0-cycle $Z_i = Z_i' \cup Z_i''$ with Z_i' is a generic 0-cycle in the Hilbert scheme $Hilb^{l(Z_i')}(B)$ (so $l(Z_i')$ has to be sufficiently large, i.e., $l(Z_i') \geq \max(p_g, h^0(B, \mathcal{O}_B(rH + K_B)))$; this also guarantees that Z_i' satisfies the Cayley-Bacharach property with respect to $|rH + K_B|$). Moreover, Z_i'' is a reduced 0-cycle (chosen suitably generic) of length $l(Z_i'') \geq 4(r-1)^2 H^2$. Then one has for a surface with $p_g(B) = 0$ (like $\mathbf{F_0} = \mathbf{P^1} \times \mathbf{P^1}$)

$$l(Z) = \sum_{i=1}^{r-1} l(Z_i') + l(Z_i'') \geq (r-1)\Big(1 + h^0(B, \mathcal{O}_B(rH + K_B)) + 4(r-1)^2 H^2\Big) \tag{B.3}$$

Applied to our situation where $H = hc_1$ and $r = 2$ one gets (for $h = 1/2$)

$$c_2(E_i) = k_i = l_2(Z) - 2$$
$$l_2(Z) \geq 2 + 8(2h-1)h + 32h^2 = 10 \tag{B.4}$$

Invariance of E_i. The involution on the base is just $\tau_B : (z_1, z_2) \longrightarrow (-z_1, -z_2)$ (written in affine coordinates). The input data in the defining extension of E_i are just divisors and point sets. Therefore it suffices to show that one can first choose these objects themselves invariant, and then to assure the existence of an invariant extension.

For divisors like $D = \mathcal{O}_{\mathbf{F}_0}(pb + qf)$ one just selects for the defining equation those monomials $z_1^r z_2^s$ with $r + s$ even. For the point sets of $l(Z) = 2k$ elements one chooses k 'mirror-pairs' of points $\{x, \tau_B(x)\}$ (for an odd number of points one adjoins a fixed point).

In the explicit construction a genericity requirement on the points arises besides the bound on the cardinality shown in (B.4). Furthermore, as just described, the invariance of the bundles will need among other requirements the invariance of the point set; i.e., the point set has to consist of mirror pairs (plus a fixed point if the cardinality is odd; the four fixed points are not generic, so we prefer to work with an even cardinality). One convinces oneself that the concrete genericity requirements needed in the construction of [27] are not violated by the fact that the point set Z consists of mirror pairs as dictated by the invariance requirement; for the required sort of genericity in these arguments is of the type 'choose from an open set' (say a complement of divisors) and does not restrict the mutual positions of the points).

For the extension note that the action of the involution τ_B breaks the space of extensions into invariant and anti-invariant parts. To guarantee the invariance of an extension we need to check that the dimension of the $+$ eigenspace is non-vanishing. For this we need to describe the appropriate action on Ext^1. Note that the τ action on the extension bundle is reflected by the τ action on $H^1(X, Hom(W, U))$. For invariant bundles W, U one can still twist the τ action on one of them by the full reflection $v \to -v$ in the fiber vector space, switching the action on Ext^1 by a sign. Therefore as soon as $Ext^1 \neq 0$ one can conclude that even an invariant extension bundle exists.

C. The fixed point formula on the moduli space

Let $\mathcal{M}^H(r, 0, k)$ be the moduli space of bundles E over the surface B (stable w.r.t. H) of rank r, $c_1(E) = 0$ and $c_2(E) = k$. According to [28] this space is non-empty if $c_2(E_i) > rk(E_i) + 1$ (provided that $p_g(B) = 0$ as in our case of $B = \mathbf{F_0}$). We want to compute the dimension of the moduli space $\mathcal{M}^H_+(r, 0, k)$ of stable τ_B-invariant bundles. Recall that $H^0(B, ad\, E) = 0$ (E is 'simple') for E stable and $H^2(B, ad\, E) = 0$ (E is 'good') at smooth points of the moduli space; one has then $\dim \mathcal{M}^H(r, 0, k) = \dim H^1(B, ad\, E)$. Recall also that on a surface B with *effective* anti-canonical divisor a simple E is already good (cf. Ch. 6, Prop. 17 [17]); in other words $\mathcal{M}^H(r, 0, k)$ (consisting of stable and therefore simple bundles) is smooth.

Assume that the action of the involution τ_B can be lifted to an action on E, so the action of τ_B lifts also to an action on the endomorphism bundle $End(E)$. The index of the $\bar{\partial}$ operator then generalizes to a character valued index where for each $g \in G$ one defines $\text{index}(g) = \sum_{i=0}^{2} (-1)^{i+1} \text{Tr}_{H^i(X, End(E))} g$. For the linear combination $\frac{1+\tau_B}{2}$ which projects onto the even subspace one finds the $\text{index}(\frac{1+\tau_B}{2}) = \sum_{i=0}^{2} (-1)^{i+1} \dim H^i_+(X, End(E))$.

More precisely, recall that the index of the $\bar{\partial}$ operator arises from the complex $(E_i) = (\mathcal{A}^{0,i}(B, E))$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
\tau_B^* E & \xrightarrow{\phi_{\tau_B}} & E \\
\downarrow & & \downarrow \\
B & \xrightarrow{\tau_B} & B
\end{array}
\tag{C.1}
$$

(Note that, on the other hand, E is called τ_B-invariant if such a bundle isomorphism exists which covers the identity in the base.)

This gives rise to corresponding maps $\phi^{(i)}_{\tau_B}$ for the spaces in the complex, and then also to maps $\Phi^{(i)}_{\tau_B} : \Gamma(E_i) \longrightarrow \Gamma(E_i)$ via the compositions $\Gamma(E_i) \xrightarrow{\phi_{\tau_B}} \Gamma(\tau_B^* E_i) \xrightarrow{\phi^{(i)}_{\tau_B}} \Gamma(E_i)$. So there are also induced maps in cohomology and we can speak about the invariant subspaces $H^i_+(B, EndE)$. What concerns the other side of the fixed point formula, one has for $b \in$ B linear maps $(\phi^{(i)}_{\tau_B})_b : (E_i)_{\tau_B(b)} \longrightarrow (E_i)_b$, so one can define in particular the contributions $\frac{\text{Tr}_{(T_b^{0,i})^* \otimes EndE} \tau_B}{\det(1 - D\tau_B)|_b}$ for the fixed points.

To compute the dimension of the submanifold $\mathcal{M}^H_+(r, 0, k) \hookrightarrow \mathcal{M}^H(r, 0, k)$, the strategy is to consider the corresponding decomposition of the tangent space of $\mathcal{M}^H(r, 0, k)$ at a point representing an invariant bundle; for the following discussion let us assume that such a point in the moduli space exists. We will compute the dimension $\dim H^1_+(B, ad\, E) = \dim \mathcal{M}^H_+(r, 0, k)$ of the tangent space to the invariant bundles, or in other words the dimension of the invariant part of the tangent space. I.e., by the mentioned unobstructedness, one can detect a non-trivial dimension of $\mathcal{M}^H_+(r, 0, k)$ (if one can secure its non-emptyness at all) by looking at invariant first-order deformations. Note that we can not apply this type of argument directly on X as there can very well be obstructions to first-order deformations.

Although we will have shown $h_+^1 > 0$, we still have to make sure that some invariant bundle E exists; the corresponding statement that (besides dimensional arguments and smoothness) the full moduli space $\mathcal{M}^H(r, 0, k)$ is non-empty was given in [28]. In other words one would have to augment the abstract existence argument given here by one concrete construction of an invariant bundle. This was given in the previous appendix.

Note that one has from $End\, E = \mathcal{O}_B \oplus ad\, E$ that $h^0(B, End\, E) = h^0(B, ad\, E) + 1$ and $h^i(B, End\, E) = h^i(B, ad\, E)$ for $i = 1, 2$.

So consider the evaluation using the Atiyah-Bott fixed point theorem

$$
\begin{aligned}
\sum_{i=0}^2 (-1)^i \dim H_+^i(B, End\, E) &= \sum_{i=0}^2 (-1)^i \operatorname{Tr}_{H^i(B, End\, E)} \frac{1 + \tau_B}{2} \\
&= \frac{1}{2} \chi(B, End\, E) + \frac{1}{2} \sum_{i=0}^2 (-1)^i \operatorname{Tr}_{H^i(B, End E)} \tau_B \\
&= \frac{1}{2} \int_B ch(E) ch(E^*) td(B) \\
&\quad + \frac{1}{2} \sum_{\tau_B(b) = b} \sum_{i=0}^2 (-1)^i \frac{\operatorname{Tr}_{(T_b^{0,i})^* \otimes End E} \tau_B}{\det(1 - D\tau_B)|_b} \\
&= \frac{1}{2}(r^2 - 2rk) + \frac{1}{2} 4 \frac{4r^2}{4}
\end{aligned}
\tag{C.2}
$$

(without the inserted τ_B action this reasoning gives the dimension $2kr - (r^2 - 1) + h^2(B, ad\, E)$ of the tangent space to the moduli space). Here $\sum (-1)^i \operatorname{Tr}_{H^i} \tau_B$ is evaluated by the fixed point theorem, giving a sum over the four fixed points weighted by a suitable determinant.

Recall that on $B = \mathbf{F_0} = \mathbf{P^1} \times \mathbf{P^1}$ the involution τ_B is given in local affine coordinates by $(z_1, z_2) \longrightarrow (-z_1, -z_2)$. This has four fixed points at $(0, 0), (0, \infty), (\infty, 0), (\infty, \infty)$. The differential being given by the diagonal matrix diag(-1, -1) one finds for the denominator contributions $\det(1 - D\tau_B)|_b = 4$. For what concerns the numerator note that

$$
\begin{aligned}
\sum_{i=0}^2 (-1)^i \operatorname{Tr}_{(T_b^{0,i})^* \otimes End E} \tau_B &= \sum_{i=0}^2 (-1)^i \operatorname{Tr}_{(T_b^{0,i})^*} \tau_B \cdot \dim End E + \dim(T_b^{0,i})^* \cdot \operatorname{Tr}_{End(E)} \tau_B \\
&= \Big(1 - (-2) + 1\Big) r^2 + \Big(1 - (+2) + 1\Big) \operatorname{Tr}_{End(E)} \tau_B
\end{aligned}
\tag{C.3}
$$

(from the local expressions $1, d\bar{z}_i, d\bar{z}_1 \wedge d\bar{z}_2$). Thus one gets for the fixed point correction term given by the double sum $4 \cdot \frac{4r^2}{4} = 4r^2$. So we find for the dimension $\dim H_+^1(B, ad\, E)$ of the space of even moduli

$$
\begin{aligned}
h^1(B, ad\, E) - 1 &= 2rk - r^2 \\
h_+^1(B, ad\, E) - 1 &= \frac{2rk - r^2}{2} - 2r^2
\end{aligned}
\tag{C.4}
$$

Therefore we get as condition for the existence of an invariant bundle

$$
c_2(E) \;\geq\; \frac{5}{2} r - \frac{1}{r}
\tag{C.5}
$$

whereas the condition for the existence of a general bundle was just $c_2(E) \geq \frac{1}{2} r - \frac{1}{2r}$.

D. Solutions without an additional U(1) but with exotic matter

In the main body of the paper we have described the solutions to the system of equations (6.1) in the case $\alpha\beta = 0$; this amounts to cancelling any exotic matter beyond the Standard Model (plus the right handed neutrino). But one is paying a price for this: the mentioned condition is also the one giving the anomaly of the unbroken U(1). Only if the latter is anomalous it becomes massive, and so decoupling from the low-energy spectrum. This decoupling is inhibited if the exotic matter is cancelled.

Alternatively one can study the case where the additional U(1) really becomes anomalous and decouples if one is willing to keep the additional exotic matter. So, to describe the corresponding solutions let us study the case $\alpha\beta \neq 0$. The symmetrical cases $p = q$ and $k_1 = k_2$ are independently excluded by (5.14) and (2.8), respectively. Further $p + q \leq 0$ and $p, q \leq 2$ from $\alpha c_1 \leq 0$ and $\alpha \leq c_1$, imply, together with $p, q \geq -2$ from $\alpha \geq -c_1$, the list

$$\alpha = (-2,-1),\ (-2,0),\ (-1,0) \qquad \text{or}\ (p \leftrightarrow q) - \text{reflections of these} \tag{D.1}$$

Though not a priori excluded, one finds by inspection of the solutions to the full system of conditions that the cases $(-2,1),(-2,2),(-1,1),(-1,2)$ do not occur.

The complete set of solutions for the instanton numbers of the two plane bundles on the base is for $\beta = (1,-1)$ given by (for $\beta = (-1,1)$ one just interchanges the k_i)

α	k_1	k_2	i
$(-2,-1)$	$9+i$	$8+i$	$i = 0,\dots,7$
$(-2,-1)$	$15+i$	$8+i$	$i = 0,\dots,4$
$(-2,0)$	$13+i$	$8+i$	$i = 0,1$
$(-1,0)$	$9+i$	$8+i$	$i = 0,\dots,3$
$(-1,0)$	$15+i$	$8+i$	$i = 0$

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